

Penrose Process in Kerr-Taub-NUT Spacetime

A.A. Abdujabbarov^{1,2} • B.J. Ahmedov^{1,2,3} •
S.R. Shaymatov^{1,2} • A.S. Rakhmatov⁴

Abstract

Penrose process on rotational energy extraction of the black hole in the Kerr-Taub-NUT spacetime is studied. It has been shown that for the radial motion of particles NUT parameter slightly shifts the shape of the effective potential down. The dependence of the extracted energy from compact object on NUT parameter has been found.

Keywords Penrose process Ergosphere Kerr-Taub-NUT spacetime

1 Introduction

At present there is no any observational evidence for the existence of gravitomagnetic monopole, i.e. so-called NUT (Newman, Unti & Tamburino – Newman et al. (1963)) parameter or *magnetic mass*. Therefore study of the motion of the test particles and energy extraction mechanisms in NUT spacetime may provide new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues (See, e.g. Nouri-Zonoz 2004; Kagramanova et al. 2008;

Morozova and Ahmedov 2009, where solutions for electromagnetic waves and interferometry in spacetime with NUT parameter have been studied.). Kerr-Taub-NUT spacetime with Maxwell and dilation fields is recently investigated by Aliev et al. (2008). In our preceding papers (Morozova et al. 2008; Abdujabbarov et al. 2008) we have studied the plasma magnetosphere around a rotating, magnetized neutron star and charged particle motion around compact objects immersed in external magnetic field in the presence of the NUT parameter.

Penrose process (see, e.g. Penrose 1969) for the extraction of energy from rotating black hole is based on the existence of negative energy orbits in the ergosphere, the region bounded by the event horizon and the static limit. Gariel et al. (2010) have examined the possibility that astrophysical jet collimation may arise from the geometry of rotating black holes and the presence of high-energy particles resulting from a Penrose process rather than from effect of the magnetic fields. Detailed study of the energetics of the Kerr-Newman black hole by the Penrose process is given by Bhat et al. (1984). Energetics of a rotating charged black hole in 5-dimensional supergravity has been recently considered by Prabhu and Dadhich (2010).

The geodesics of the Kerr-Taub-NUT spacetime share many of the properties of the ergosphere in the field of a magnetic monopole. A thorough discussion and comparison of these orbits can be found in papers (Zimmerman and Shahir 1989; Kagramanova et al. 2010).

Here we first study motion of the test particles around rotating compact object with nonvanishing NUT parameter. The effective potential of the radial motion of the test particles around rotating compact object is investigated in the presence of the NUT parameter using the Lagrangian formalism. Finally we study the energy extraction mechanism from compact

A.A. Abdujabbarov

B.J. Ahmedov

S.R. Shaymatov

A.S. Rakhmatov

¹Institute of Nuclear Physics, Ulughbek, Tashkent 100214, Uzbekistan

²Ulugh Begh Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan

³International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

⁴National University of Uzbekistan, Tashkent 100174, Uzbekistan

object through Penrose process in the Kerr-Taub-NUT spacetime.

The outline of the paper is as follow. In the Sec. 2 we study the ergosphere and motion of the test particles in the Kerr-Taub-NUT spacetime. The Sec. 3 is devoted to study the energy extraction mechanisms through Penrose process in the Kerr-Taub-NUT spacetime. The concluding remarks are given in the Sec. 4.

Throughout the paper, we use a space-like signature $(-, +, +, +)$ and a system of units in which $G = 1 = c$ (However, for those expressions with an astrophysical application we have written the speed of light explicitly.). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2 Ergosphere around compact object in Kerr-Taub-NUT spacetime

We consider electromagnetic fields of compact astrophysical objects in Keer-Taub-NUT spacetime which in a spherical coordinates (ct, r, θ, ϕ) is described by the metric (see Dadhich and Turakulov 2002; Bini et al. 2003)

$$\begin{aligned} ds^2 = & -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ & + \frac{2}{\Sigma} [\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta] dt d\varphi \\ & + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] d\varphi^2, \end{aligned} \quad (1)$$

where parameters Σ , Δ and χ are defined by

$$\begin{aligned} \Sigma &= r^2 + (l + a \cos \theta)^2, \\ \Delta &= r^2 - 2Mr - l^2 + a^2, \\ \chi &= a \sin^2 \theta - 2l \cos \theta, \end{aligned}$$

M is the total mass, a is the specific angular momentum, and l is the NUT parameter of the central object.

The spacetime (1) has a horizon where the 4 velocity of a co-rotating observer turns to null, or the surface $r = \text{const}$ becomes null:

$$r_+ = M + \sqrt{M^2 + l^2 - a^2}. \quad (2)$$

The static limit is defined where the time-translation Killing vector $\xi_{(t)} \equiv \partial/\partial t$ becomes null (i.e. $g_{00} = 0$) and static limit of the black hole can be described as

$$r_{st} = M + \sqrt{M^2 + l^2 - a^2 \cos^2 \theta}. \quad (3)$$

Considering only the outer horizon, r_+ and static limit, r_{st} , it can be verified that the static limit always

lies outside the horizon. The region between the two is called the ergosphere, where timelike geodesics cannot remain static but can remain stationary due to corotation with the BH with the specific frame dragging angular velocity at the given location in the ergosphere. This is the region of spacetime where timelike particles with negative angular momentum relative to the BH can have negative energy relative to the infinity.

In Fig.1 the dependence of the shape of the ergosphere from small dimensionless parameter $\tilde{l} = l/M$ is shown. From the dependence one can easily see that in the presence of the NUT parameter radius of the event horizon becomes larger. However a relative volume of the ergosphere is decreased.

Due to the existence of an ergosphere around the black hole, it is possible to extract energy from it by means of the Penrose process. Inside the ergosphere, it is possible to have a timelike or null trajectory with negative total energy. As a result, shoot a small particle A into the ergosphere from outside with energy at infinity. When the particle is deep down near the horizon, let it to explode into two parts, B and C , one of which attains negative energy relative to infinity and falls down the hole but other part escapes back to radial infinity by conservation of energy with energy greater than that of the original incident particle. This is how the energy could be extracted from the hole by axial accretion of particles with the suitable angular momentum and l parameters. Consider the equation of motion of such negative energy particle. The Lagrangian for this particle can be written as:

$$\begin{aligned} 2\mathcal{L} = & -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) \dot{t}^2 + \frac{\Sigma}{\Delta} \dot{r}^2 + \Sigma \dot{\theta}^2 \\ & + \frac{2}{\Sigma} [\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta] \dot{t} \dot{\varphi} \\ & + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] \dot{\varphi}^2, \end{aligned} \quad (4)$$

and according to (4) the generalized momenta are given by

$$\begin{aligned} p_t = & -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) \dot{t} \\ & + \frac{1}{\Sigma} (\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta) \dot{\varphi} = E, \end{aligned} \quad (5)$$

$$\begin{aligned} -p_\varphi = & -\frac{1}{\Sigma} (\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta) \dot{t} \\ & - \frac{1}{\Sigma} ((\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta) \dot{\varphi} = L, \end{aligned} \quad (6)$$

$$-p_r = -\frac{\Sigma}{\Delta} \dot{r}, \quad (7)$$

$$-p_\theta = -\Sigma \dot{\theta}, \quad (8)$$

where superior dots denote differentiation with respect to an affine parameter τ . (The conservation of p_t and

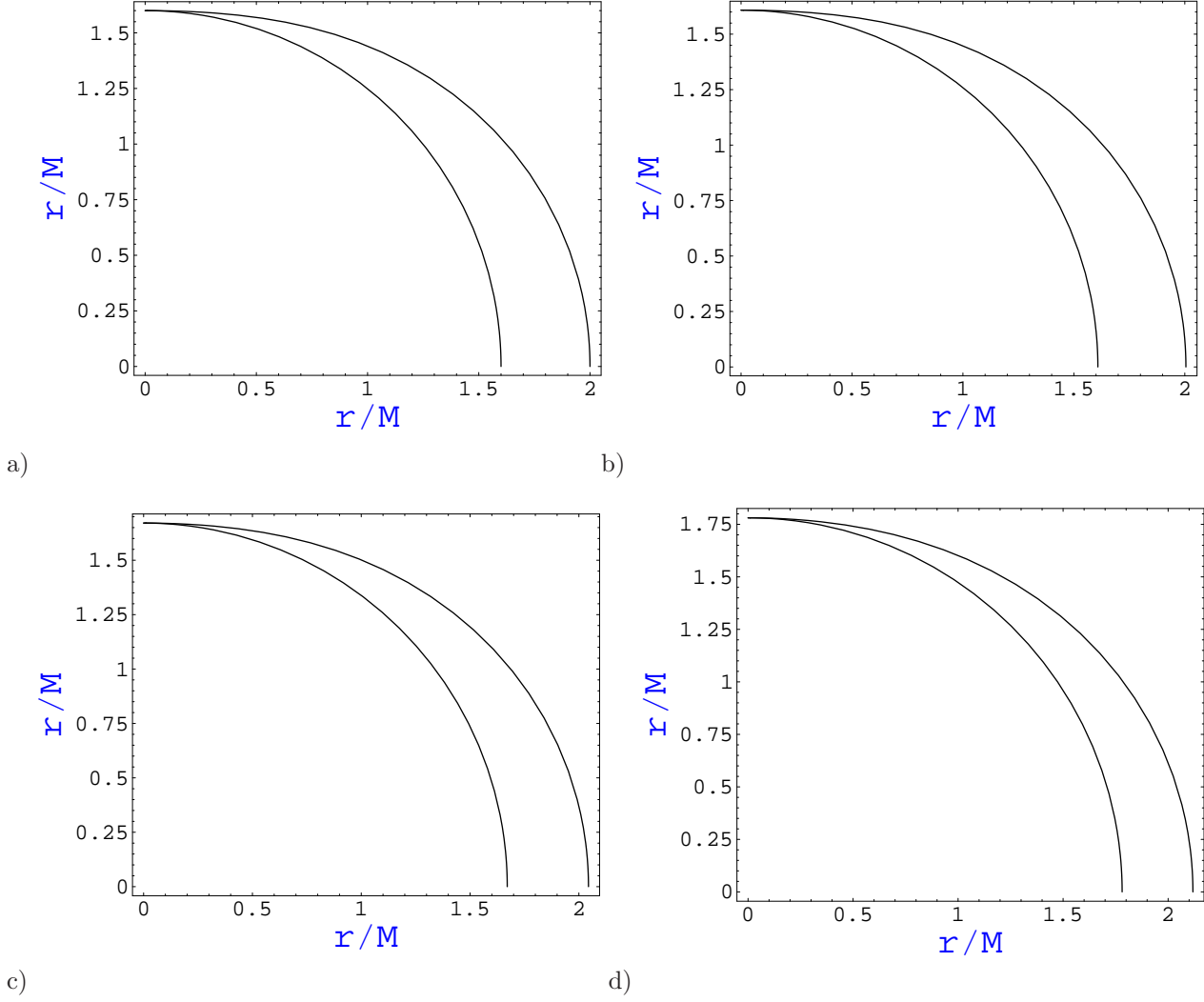


Fig. 1 The dependence of the shape of the ergosphere from the small dimensionless NUT parameter \tilde{l} : a) $\tilde{l} = 0$, b) $\tilde{l} = 0.1$ c) $\tilde{l} = 0.3$ d) $\tilde{l} = 0.5$.

p_φ follows from the independence of the Lagrangian on t and φ which, in turn, is a manifestation of the stationary and the axisymmetric character of the Kerr-Taub-NUT geometry.)

The Hamiltonian for the test particle in spacetime (1) is given by

$$\begin{aligned}\mathcal{H} = & p_t \dot{t} + p_\varphi \dot{\varphi} + p_r \dot{r} + p_\theta \dot{\theta} - \mathcal{L} \\ & - \frac{1}{2\Sigma} (\Delta - a^2 \sin^2 \theta) \dot{t}^2 + \frac{\Sigma}{2\Delta} \dot{r}^2 + \frac{1}{2} \Sigma \dot{\theta}^2 \\ & + \frac{1}{\Sigma} [\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta] \dot{t} \dot{\varphi} \\ & + \frac{1}{2\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] \dot{\varphi}^2,\end{aligned}\quad (9)$$

and from the independence of the Hamiltonian on t , one can get

$$\begin{aligned}2\mathcal{H} = & \frac{\Sigma}{\Delta} \dot{r}^2 + \Sigma \dot{\theta}^2 + \left[-\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) \dot{t} \right. \\ & + \frac{1}{\Sigma} (\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta) \dot{\varphi} \left. \right] \dot{t} \\ & + \left[\frac{1}{\Sigma} (\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta) \dot{t} \right. \\ & + \frac{1}{\Sigma} ((\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta) \dot{\varphi} \left. \right] \dot{\varphi} \\ = & E\dot{t} - L\dot{\varphi} + \frac{\Sigma}{\Delta} \dot{r}^2 + \Sigma \dot{\theta}^2 = \delta = \text{const.}\end{aligned}\quad (10)$$

We may, without loss of generality, set

$$\delta = \begin{cases} 1 & \text{for time-like geodesics,} \\ 0 & \text{for null geodesics.} \end{cases}\quad (11)$$

Then from the expressions (5) and (6) one can easily obtain the expressions for $\dot{\varphi}$ and \dot{t} as

$$\dot{\varphi} = \frac{1}{\Delta} \left[\frac{(\Delta - a^2 \sin^2 \theta)(\chi E - L) - a\Sigma E \sin^2 \theta}{\Sigma \sin^2 \theta} \right],\quad (12)$$

$$\begin{aligned}\dot{t} = & -\frac{1}{\Delta} \left[\frac{\Delta\Sigma L}{\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta} \right. \\ & + \frac{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}{\Sigma (\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta)} \\ & \times ((\Delta - a^2 \sin^2 \theta)(\chi E - L) - a\Sigma E \sin^2 \theta) \left. \right].\end{aligned}\quad (13)$$

It was first shown by Zimmerman and Shahir (1989) for spherical symmetric case (NUT spacetime) and later by Bini et al. (2003) for axial symmetric case (Kerr-Taub-NUT spacetime) that the orbits of the test particles are confined to a cone with the opening angle θ given by $\cos \theta = 2El/L$. It also follows that in this case the equations of motion on the cone depend

on l only via l^2 (Bini et al. 2003; Abdujabbarov et al. 2008). The main point is that the small value for the upper limit for gravitomagnetic moment has been obtained by comparing theoretical results with experimental data as (i) $l \leq 10^{-24}$ from the gravitational microlensing (Rahvar and Habibi 2004), (ii) $l \leq 1.5 \cdot 10^{-18}$ from the interferometry experiments on ultra-cold atoms (Morozova and Ahmedov 2009), (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer (Kagramanova et al. 2008). Due to the smallness of the gravitomagnetic charge let us consider the motion in the quasi-equatorial plane when the motion in θ direction changes as $\theta = \pi/2 + \delta\theta(t)$, where $\delta\theta(t)$ is the term of first order in l , then it is easy to expand the trigonometric functions as $\sin \theta = 1 - \delta\theta^2(t)/2 + \mathcal{O}(\delta\theta^4(t))$ and $\cos \theta = \delta\theta(t) - \mathcal{O}(\delta\theta^3(t))$. Now inserting the expressions (12) and (13) to the equation (10), and neglecting the small terms $\mathcal{O}(\delta\theta^2(t))$, one can easily obtain equation for the radial motion as follows:

$$\begin{aligned}\Sigma \dot{r}^2 = & 2(L - aE)^2 \frac{(Mr + l^2)}{r^2 + l^2} \\ & + E^2(r^2 + a^2 + l^2) - L^2 - \delta\Delta.\end{aligned}\quad (14)$$

As we have noted, $\delta = 0$ for null geodesics and equation for radial motion (14) becomes

$$\Sigma \dot{r}^2 = 2(L - aE)^2 \frac{(Mr + l^2)}{r^2 + l^2} + E^2(r^2 + a^2 + l^2) - L^2. \quad (15)$$

Hereafter, it is more convenient to distinguish the geodesics by the impact parameter $D = L/E$. First, we study the geodesics with the impact parameter

$$D = a, \quad \text{when} \quad L = aE, \quad (16)$$

and equations (12), (13) and (15) reduce to

$$\dot{r} = \pm E, \quad (17)$$

$$\dot{t} = -(r^2 + a^2 + l^2) \frac{E}{\Delta}, \quad (18)$$

$$\dot{\varphi} = -\frac{aE}{\Delta}. \quad (19)$$

The radial coordinate is uniformly described with respect to the affine parameter while the equations governing t and φ are

$$\frac{dt}{dr} = \pm \frac{r^2 + a^2 + l^2}{\Delta}, \quad (20)$$

$$\frac{d\varphi}{dr} = \pm \frac{a}{\Delta}. \quad (21)$$

3 Energy extraction by Penrose process for Kerr-Taub-NUT spacetime

Let us continue our assumption that the deflection in θ direction is reasonably small and orbits of the particles are in the quasi-equatorial plane $\theta = \pi/2 + \delta\theta(t)$. Using the assumptions mentioned in previous section one can easily rewrite the expressions (5) and (6) in the approximation $\mathcal{O}(\delta\theta^2(t))$ as a quadratic equation in energy:

$$\alpha E^2 - 2\beta E + \gamma + \frac{\Sigma}{\Delta}(p^r)^2 + \Sigma(p^\theta)^2 + m^2 = 0, \quad (22)$$

where we have used the following notations

$$\alpha = - \left(1 - 2 \frac{Mr + l^2}{r^2 + l^2} \right), \quad (23)$$

$$\beta = 4a \frac{Mr + l^2}{r^2 + l^2} L, \quad (24)$$

$$\gamma = \left(r^2 + a^2 + l^2 + 2a^2 \frac{Mr + l^2}{r^2 + l^2} \right) L^2. \quad (25)$$

From the equation (22) one can easily obtain the equation of radial motion in the following form:

$$\dot{r}^2 = E^2 - V_{\text{eff}}, \quad (26)$$

and the notation

$$V_{\text{eff}} = E^2 - 2 \frac{(Mr + l^2)(L - aE)^2}{\Sigma(r^2 + l^2)} - \frac{E^2(r^2 + a^2 + l^2)}{\Sigma} + \frac{L^2 + \Delta}{\Sigma} \quad (27)$$

denotes the effective potential of the radial motion of the test particle around rotating compact object with nonvanishing NUT parameter.

In the Fig. 2 the radial dependence of the effective potential of radial motion of the massive test particle has been shown for the different values of the dimensionless parameter \tilde{l} . Here for the energy and momenta of the particle the following values are taken: $E/m = 0.9$, $L/mM = 4.3$. The presence of the parameter l slightly shifts the shape of the effective potential down.

Now, as the particle falls through the horizon, the mass of the black hole will change by $\delta M = E$. There is no upper limit for change of mass of the central black hole. The infalling big number of particles with positive energy can essentially increase the mass of the black hole. But there is a lower limit on δM which could be added to the black hole corresponding to $m = 0$, $p^\theta = 0$ and $p^r = 0$. Evaluating all of the required quantities at the horizon $r = r_+$, we get the limit for the change in black hole mass as

$$\delta M = - \frac{L_z a (Mr_+ + l^2)}{r_+^2 - 2Mr_+} D, \quad (28)$$

where

$$D = 4 \left[1 - \left\{ 1 + \left(1 - \frac{a^2(r_+^2 + 2Mr_+ + 3l^2)}{(r_+^2 + l^2)^2} \right) \frac{(r_+^2 - 2Mr_+ - l^2)}{16a^2} \left(\frac{Mr_+ + l^2}{r_+^2 + l^2} \right)^{-2} \right\}^{\frac{1}{2}} \right]. \quad (29)$$

To be able to extract energy from the black hole ($\delta M < 0$), we must therefore have

$$\frac{L_z a (Mr_+ + l^2)}{r_+^2 - 2Mr_+} D > 0. \quad (30)$$

In the Fig. 3 the dependence of the extracted energy from the black hole on the small dimensionless parameter \tilde{l} has been shown: with increasing the parameter \tilde{l} the relative extraction of the energy becomes more stronger. The graph shows that the extraction of the energy is directly proportional to the parameter \tilde{l} , that is with increasing \tilde{l} , the extracted energy also increases.

4 Conclusion

We have studied the properties of the ergosphere of the black hole in the Kerr-Taub-NUT spacetime. The dependence of the shape of the ergosphere from small dimensionless NUT parameter shows that the radius of the event horizon becomes larger. However the relative volume of the ergosphere is decreased and the extracted energy from the black hole is raised up by the small dimensionless parameter \tilde{l} .

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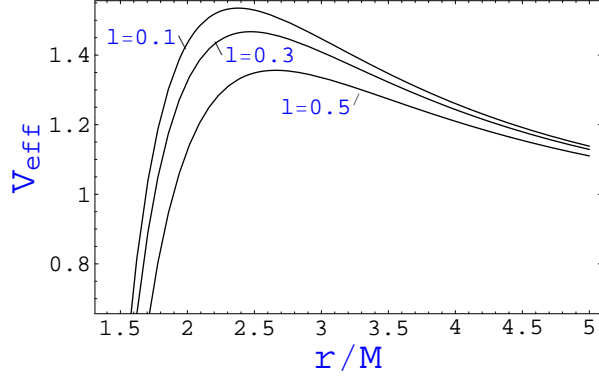


Fig. 2 The radial dependence of the effective potential of radial motion of the particle for the different values of the dimensionless NUT parameter \tilde{l} .

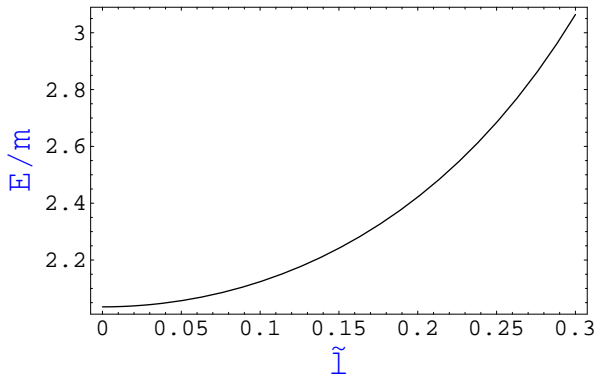


Fig. 3 The dependence of the extracted energy from the black hole on the dimensionless NUT parameter \tilde{l} .

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